Lecture 34: MON 20 APR

Electrical Oscillations, LC Circuits, Alternating Current II

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Damped LCR Oscillator

Ideal LC circuit without resistance: oscillations go on forever; \( \omega = (LC)^{-1/2} \)

Real circuit has resistance, dissipates energy: oscillations die out, or are “damped”

Math is complicated! Important points:

- Frequency of oscillator shifts away from \( \omega = (LC)^{-1/2} \)
- Peak CHARGE decays with time constant = \( \tau_{QLCR} = 2L/R \)
- For small damping, peak ENERGY decays with time constant \( \tau_{ULCR} = L/R \)

\[
U_{\text{max}} = \frac{Q^2}{2C} e^{-\frac{Rt}{L}}
\]
31.2.1. Which one of the following choices will damp oscillations in an LC circuit?

a) increase the inductance

b) increase the emf

c) increase the circuit resistance

d) increase the capacitance
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Damped Oscillations in an $RCL$ Circuit

If we add a resistor in an $RL$ circuit (see figure) we must modify the energy equation, because now energy is being dissipated on the resistor: $\frac{dU}{dt} = -i^2 R$.

$$U = U_E + U_B = \frac{q^2}{2C} + \frac{Li^2}{2} \quad \rightarrow \quad \frac{dU}{dt} = \frac{q}{C} \frac{dq}{dt} + Li \frac{di}{dt} = -i^2 R$$

$$i = \frac{dq}{dt} \quad \rightarrow \quad \frac{di}{dt} = \frac{d^2 q}{dt^2} \quad \rightarrow \quad L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = 0.$$  This is the same equation as that of the damped harmonics oscillator: $m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0$, which has the solution:

$$x(t) = x_m e^{-bt/2m} \cos(\omega' t + \phi).$$  The angular frequency $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$.

For the damped $RCL$ circuit the solution is:

$$q(t) = Q e^{-Rt/2L} \cos(\omega' t + \phi).$$  The angular frequency $\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$.  (31-6)
The equations above describe a harmonic oscillator with an exponentially decaying amplitude \( Q e^{-Rt/2L} \). The angular frequency of the damped oscillator

\[
\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}
\]

is always smaller than the angular frequency \( \omega = \sqrt{\frac{1}{LC}} \) of the undamped oscillator. If the term \( \frac{R^2}{4L^2} \ll \frac{1}{LC} \) we can use the approximation \( \omega' \approx \omega \).

\[
\tau_{RC} = RC \quad \tau_{RL} = \frac{L}{R} \quad \tau_{RCL} = 2L/R
\]
Damped RLC circuit: charge amplitude

A series RLC circuit has inductance $L = 12$ mH, capacitance $C = 1.6 \, \mu F$, and resistance $R = 1.5 \, \Omega$ and begins to oscillate at time $t = 0$.

(a) At what time $t$ will the amplitude of the charge oscillations in the circuit be 50% of its initial value? (Note that we do not know that initial value.)

$$R^2 \ll \frac{1}{4L^2} \quad \text{and} \quad 10^3 \ll 10^8! \quad \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \approx \sqrt{\frac{1}{LC}} = \omega$$

Solving for $t$ and then substituting given data yield

$$t = -\frac{2L}{R} \ln 0.50 = -\frac{(2)(12 \times 10^{-3} \, \text{H})(\ln 0.50)}{1.5 \, \Omega}$$

$$= 0.0111 \, \text{s} \approx 11 \, \text{ms}.$$  (Answer)

(b) How many oscillations are completed within this time?

**KEY IDEA**

The amplitude of the charge oscillations decreases exponentially with time $t$. According to Eq. 31-25, the charge amplitude at any time $t$ is $Qe^{-Rt/2L}$, in which $Q$ is the amplitude at time $t = 0$.

**Calculations:** We want the time when the charge amplitude has decreased to 0.50$Q$, that is, when

$$Qe^{-Rt/2L} = 0.50Q.$$

We can now cancel $Q$ (which also means that we can answer the question without knowing the initial charge). Taking the natural logarithms of both sides (to eliminate the exponential function), we have

$$-\frac{Rt}{2L} = \ln 0.50.$$

The time for one complete oscillation is the period $T = \frac{2\pi}{\omega}$, where the angular frequency for $LC$ oscillations is given by Eq. 31-4 ($\omega = 1/\sqrt{LC}$).

**Calculation:** In the time interval $\Delta t = 0.0111 \, \text{s}$, the number of complete oscillations is

$$\frac{\Delta t}{T} = \frac{\Delta t}{2\pi \sqrt{LC}}$$

$$= \frac{0.0111 \, \text{s}}{2\pi[(12 \times 10^{-3} \, \text{H})(1.6 \times 10^{-6} \, \text{F})]^{1/2}} \approx 13.$$  (Answer)

Thus, the amplitude decays by 50% in about 13 complete oscillations. This damping is less severe than that shown in Fig. 31-3, where the amplitude decays by a little more than 50% in one oscillation.
Calculation: In the time interval $\Delta t = 0.0111$ s, the number of complete oscillations is

$$\frac{\Delta t}{T} = \frac{\Delta t}{2\pi \sqrt{LC}}$$

$$= \frac{0.0111 \text{ s}}{2\pi[(12 \times 10^{-3} \text{ H})(1.6 \times 10^{-6} \text{ F})]^{1/2}} \approx 13.$$  

(Answer)

Thus, the amplitude decays by 50% in about 13 complete oscillations. This damping is less severe than that shown in Fig. 31-3, where the amplitude decays by a little more than 50% in one oscillation.
Summary

• Capacitor and inductor combination produces an electrical oscillator, natural frequency of oscillator is $\omega = 1/\sqrt{LC}$

• Total energy in circuit is conserved: switches between capacitor (electric field) and inductor (magnetic field).

• If a resistor is included in the circuit, the total energy decays (is dissipated by $R$).
Alternating Current:

To keep oscillations going we need to drive the circuit with an external emf that produces a current that goes back and forth.

Notice that there are two frequencies involved: $\omega$ at which the circuit would oscillate “naturally”. The other is $\omega_d$ the driving frequency at which we drive the oscillation.

However, the “natural” oscillation usually dies off quickly (exponentially) with time. Therefore in the long run, circuits actually oscillate with the frequency at which they are driven. (All this is true for the gentleman trying to make the lady swing back and forth in the picture too).
We have studied that a loop of wire, spinning in a constant magnetic field will have an induced emf that oscillates with time,

\[ \mathcal{E} = \mathcal{E}_m \sin(\omega_d t) \]

That is, it is an AC generator.

AC’s are very easy to generate, they are also easy to amplify and decrease in voltage. This in turn makes them easy to send in distribution grids like the ones that power our homes.

Because the interplay of AC and oscillating circuits can be quite complex, we will start by steps, studying how currents and voltages respond in various simple circuits to AC’s.
31.6: Forced Oscillations:

Fig. 31-7 A single-loop circuit containing a resistor, a capacitor, and an inductor. A generator, represented by a sine wave in a circle, produces an alternating emf that establishes an alternating current; the directions of the emf and current are indicated here at only one instant.

Whatever the natural angular frequency $\omega$ of a circuit may be, forced oscillations of charge, current, and potential difference in the circuit always occur at the driving angular frequency $\omega_d$. 
31.9: The Series RLC Circuit, Resonance:

Fig. 31-16  Resonance curves for the driven RLC circuit of Fig. 31-7 with $L = 100 \, \mu\text{H}$, $C = 100 \, \text{pF}$, and three values of $R$. The current amplitude $I$ of the alternating current depends on how close the driving angular frequency $\omega_d$ is to the natural angular frequency $\omega$. The horizontal arrow on each curve measures the curve’s half-width, which is the width at the half-maximum level and is a measure of the sharpness of the resonance. To the left of $\omega_d/\omega = 1.00$, the circuit is mainly capacitive, with $X_C > X_L$; to the right, it is mainly inductive, with $X_L > X_C$. 

- **Driving $\omega_d$ equal to natural $\omega$**
  - high current amplitude
  - circuit is in resonance
  - equally capacitive and inductive
  - $X_C$ equals $X_L$
  - current and emf in phase
  - zero $\phi$

- **Low driving $\omega_d$**
  - low current amplitude
  - ICE side of the curve
  - more capacitive
  - $X_C$ is greater
  - current leads emf
  - negative $\phi$

- **High driving $\omega_d$**
  - low current amplitude
  - ELI side of the curve
  - more inductive
  - $X_L$ is greater
  - current lags emf
  - positive $\phi$
Example 1: Tuning a Radio Receiver
Driven RLC With EMF Antennal

The inductor and capacitor in a car radio have one program at $L = 1$ mH & $C = 3.18$ pF. Which is the FM station?

WRKF 89.3

What is the wavelength of the radio wave from the tower?

FM radio stations: frequency is in MHz.

$$\omega = \frac{1}{\sqrt{LC}}$$

$$= \frac{1}{\sqrt{1 \times 10^{-6} \times 3.18 \times 10^{-12}}} \text{rad/s}$$

$$= 5.61 \times 10^8 \text{rad/s}$$

$$f = \frac{\omega}{2\pi}$$

$$= 8.93 \times 10^7 \text{Hz}$$

$$= 89.3 \text{ MHz}$$
Example 1: Tuning a Radio Receiver
Driven RLC at Resonance With EMF Antenna

Antenna

10 kHz bandwidth from 540-1600 kHz for 106 possible bands

AM Radio

Figure 11
31.6: Forced Oscillations:

![Diagram of a single-loop circuit containing a resistor, a capacitor, and an inductor. A generator, represented by a sine wave in a circle, produces an alternating emf that establishes an alternating current; the directions of the emf and current are indicated here at only one instant.]

Whatever the natural angular frequency $\omega$ of a circuit may be, forced oscillations of charge, current, and potential difference in the circuit always occur at the driving angular frequency $\omega_d$.

In the power grid we make sure that all circuits are far away from any resonances!
AC Driven Circuits:

1) A Resistor:

\[ emf - v_R = 0 \]

\[ v_R = emf = E_m \sin(\omega_d t) \]

\[ i_R = \frac{v_R}{R} = \frac{E_m}{R} \sin(\omega_d t) \]

Resistors behave in AC very much as in DC, current and voltage are proportional (as functions of time in the case of AC), that is, they are “in phase”.

For time dependent periodic situations it is useful to represent magnitudes using Steinmetz “phasors”. These are vectors that rotate at a frequency \( \omega_d \), their magnitude is equal to the amplitude of the quantity in question and their projection on the vertical axis represents the instantaneous value of the quantity.
AC Driven Circuits:

2) Capacitors:

\[
v_C = \text{emf} = \mathcal{E}_m \sin(\omega_d t)
\]
\[
q_C = C \text{emf} = C \mathcal{E}_m \sin(\omega_d t)
\]
\[
i_C = \frac{dq_C}{dt} = \omega_d C \mathcal{E}_m \cos(\omega_d t)
\]
\[
i_C = \omega_d C \mathcal{E}_m \sin(\omega_d t + 90^0)
\]

\[
i_C = \frac{\mathcal{E}_m}{X} \sin(\omega_d t + 90^0)
\]

where \( X = \frac{1}{\omega_d C} \) "reactance"

\[
i_m = \frac{\mathcal{E}_m}{X}
\]

looks like \( i = \frac{V}{R} \)

Capacitors "oppose a resistance" to AC (reactance) of frequency-dependent magnitude \( 1/\omega_d C \)

(this idea is true only for maximum amplitudes, the instantaneous story is more complex).
AC Driven Circuits:

3) Inductors:

\[ v_L = \text{emf} = \mathcal{E}_m \sin(\omega_d t) \]

\[ v_L = L \frac{di_L}{dt} \Rightarrow i_L = \frac{\int v_L \, dt}{L} \]

\[ i_L = -\frac{\mathcal{E}_m}{L \omega_d} \cos(\omega_d t) = \frac{\mathcal{E}_m}{L \omega_d} \sin(\omega_d t - 90^0) \]

\[ i_L = \frac{\mathcal{E}_m}{X} \sin(\omega_d t - 90^0) \]

\[ i_m = \frac{\mathcal{E}_m}{X} \quad \text{where} \quad X = L \omega_d \]

Inductors “oppose a resistance” to AC (reactance) of frequency-dependent magnitude \( w_d L \) (this idea is true only for maximum amplitudes, the instantaneous story is more complex).
Energy Transmission Requirements

The resistance of the power line $R = \frac{\rho \ell}{A}$. $R$ is fixed (220 $\Omega$ in our example).

Heating of power lines $P_{\text{heat}} = I_{\text{rms}}^2 R$. This parameter is also fixed (55 MW in our example).

Power transmitted $P_{\text{trans}} = \mathcal{E}_{\text{rms}} I_{\text{rms}}$ (368 MW in our example).

In our example $P_{\text{heat}}$ is almost 15% of $P_{\text{trans}}$ and is acceptable.

To keep $P_{\text{heat}}$ small we must keep $I_{\text{rms}}$ as low as possible. The only way to accomplish this is by increasing $\mathcal{E}_{\text{rms}}$. In our example $\mathcal{E}_{\text{rms}} = 735$ kV. To do that we need a device that can change the amplitude of any ac voltage (either increase or decrease).
Thomas Edison pushed for the development of a DC power network.

George Westinghouse backed Tesla’s development of an AC power network.

Nikola Tesla was instrumental in developing AC networks.

Edison was a brute-force experimenter, but was no mathematician. **AC cannot be properly understood or exploited without a substantial understanding of mathematics and mathematical physics**, which Tesla possessed.
The most common example is the Tesla three-phase power system used for industrial applications and for power transmission. The most obvious advantage of three phase power transmission using three wires, as compared to single phase power transmission over two wires, is that the power transmitted in the three phase system is the voltage multiplied by the current in each wire times the square root of three (approximately 1.73). The power transmitted by the single phase system is simply the voltage multiplied by the current. Thus the three phase system transmits 73% more power but uses only 50% more wire.
Against General Electric and Edison's proposal, Westinghouse, using Tesla's AC system, won the international Niagara Falls Commission contract. Tesla's three-phase AC transmission became the World's power-grid standard.

Transforming DC power from one voltage to another was difficult and expensive due to the need for a large spinning rotary converter or motor-generator set, whereas with AC the voltage changes can be done with simple and efficient transformer coils that have no moving parts and require no maintenance. This was the key to the success of the AC system. Modern transmission grids regularly use AC voltages up to 765,000 volts.
The Transformer
The transformer is a device that can change the voltage amplitude of any ac signal. It consists of two coils with a different number of turns wound around a common iron core.

The coil on which we apply the voltage to be changed is called the "primary" and it has $N_p$ turns. The transformer output appears on the second coil, which is known as the "secondary" and has $N_s$ turns. The role of the iron core is to ensure that the magnetic field lines from one coil also pass through the second. We assume that if voltage equal to $V_p$ is applied across the primary then a voltage $V_s$ appears on the secondary coil. We also assume that the magnetic field through both coils is equal to $B$ and that the iron core has cross-sectional area $A$. The magnetic flux through the primary $\Phi_p = N_pBA \rightarrow V_p = -\frac{d\Phi_p}{dt} = -N_pA\frac{dB}{dt}$ (eq. 1).

The flux through the secondary $\Phi_s = N_sBA \rightarrow V_s = -\frac{d\Phi_s}{dt} = -N_sA\frac{dB}{dt}$ (eq. 2).

(31-25)
If we divide equation 2 by equation 1 we get:

\[
\frac{V_S}{V_P} = \frac{-N_S A \frac{dB}{dt}}{-N_P A \frac{dB}{dt}} = \frac{N_S}{N_P} \rightarrow \frac{V_S}{V_P} = \frac{N_S}{N_P}.
\]  

The voltage on the secondary \( V_S = V_P \frac{N_S}{N_P} \).

If \( N_S > N_P \rightarrow \frac{N_S}{N_P} > 1 \rightarrow V_S > V_P \), we have what is known as a "step up" transformer.

If \( N_S < N_P \rightarrow \frac{N_S}{N_P} < 1 \rightarrow V_S < V_P \), we have what is known as a "step down" transformer.

Both types of transformers are used in the transport of electric power over large distances.
\[
\frac{V_S}{N_S} = \frac{V_P}{N_P}
\]
\[I_S N_S = I_P N_P\]

We have that:
\[
\frac{V_S}{N_S} = \frac{V_P}{N_P}
\]
\[→ V_S N_P = V_P N_S \quad \text{(eq. 1)}\]

If we close switch S in the figure we have in addition to the primary current \(I_P\) a current \(I_S\) in the secondary coil. We assume that the transformer is "ideal," i.e., it suffers no losses due to heating. Then we have:
\[V_P I_P = V_S I_S \quad \text{(eq. 2)}\]

If we divide eq. 2 with eq. 1 we get:
\[
\frac{V_P I_P}{V_S N_P} = \frac{V_S I_S}{V_S N_P} \quad → I_P N_P = I_S N_S.
\]

\[I_S = \frac{N_P}{N_S} I_P\]

In a step-up transformer \((N_S > N_P)\) we have that \(I_S < I_P\).

In a step-down transformer \((N_S < N_P)\) we have that \(I_S > I_P\).

(31-27)
Example, Transformer:

A transformer on a utility pole operates at \( V_p = 8.5 \text{ kV} \) on the primary side and supplies electrical energy to a number of nearby houses at \( V_s = 120 \text{ V} \), both quantities being rms values. Assume an ideal step-down transformer, a purely resistive load, and a power factor of unity.

(a) What is the turns ratio \( N_p/N_s \) of the transformer?

**KEY IDEA**

The turns ratio \( N_p/N_s \) is related to the (given) rms primary and secondary voltages via Eq. 31-79 (\( V_s = V_p N_s/N_p \)).

**Calculation:** We can write Eq. 31-79 as

\[
\frac{V_s}{V_p} = \frac{N_s}{N_p}.
\]

(Note that the right side of this equation is the inverse of the turns ratio.) Inverting both sides of Eq. 31-83 gives us

\[
\frac{N_p}{N_s} = \frac{V_p}{V_s} = \frac{8.5 \times 10^3 \text{ V}}{120 \text{ V}} = 70.83 \approx 71.
\]

(b) The average rate of energy consumption (or dissipation) in the houses served by the transformer is 78 kW. What are the rms currents in the primary and secondary of the transformer?

**Calculations:** In the primary circuit, with \( V_p = 8.5 \text{ kV} \), Eq. 31-77 yields

\[
I_p = \frac{P_{\text{avg}}}{V_p} = \frac{78 \times 10^3 \text{ W}}{8.5 \times 10^3 \text{ V}} = 9.176 \text{ A} \approx 9.2 \text{ A}.
\]

(Answer)

Similarly, in the secondary circuit,

\[
I_s = \frac{P_{\text{avg}}}{V_s} = \frac{78 \times 10^3 \text{ W}}{120 \text{ V}} = 650 \text{ A}.
\]

(Answer)

You can check that \( I_s = I_p (N_p/N_s) \) as required by Eq. 31-80.

(c) What is the resistive load \( R_s \) in the secondary circuit? What is the corresponding resistive load \( R_p \) in the primary circuit?

\[
R_s = \frac{V_s}{I_s} = \frac{120 \text{ V}}{650 \text{ A}} = 0.1846 \Omega \approx 0.18 \Omega.
\]

(Answer)

Similarly, for the primary circuit we find

\[
R_p = \frac{V_p}{I_p} = \frac{8.5 \times 10^3 \text{ V}}{9.176 \text{ A}} = 926 \Omega \approx 930 \Omega.
\]

(Answer)
31.11.2. The ac adapter for a laptop computer contains a transformer. The input of the adapter is the 120 volts from the ac wall outlet. The output from the transformer is 20 volts. What is the *turns ratio* of the transformer?

a) 0.17  
b) 6  
c) 100  
d) This cannot be determined without knowing how many turns one of the coils in the transformer has.

The turns ratio $N_p/N_s$ is related to the (given) rms primary and secondary voltages via Eq. 31-79 ($V_s = V_p N_s/N_p$).

**Calculation:** We can write Eq. 31-79 as

$$\frac{V_s}{V_p} = \frac{N_s}{N_p}.$$  \hspace{1cm} (31-83)

(Note that the right side of this equation is the *inverse* of the turns ratio.) Inverting both sides of Eq. 31-83 gives us

$$\frac{N_p}{N_s} = \frac{V_p}{V_s} = \frac{8.5 \times 10^3 \text{ V}}{120 \text{ V}} = 70.83 \approx 71.$$ \hspace{1cm} (Answer)
31.11.2. The ac adapter for a laptop computer contains a transformer. The input of the adapter is the 120 volts from the ac wall outlet. The output from the transformer is 20 volts. What is the *turns ratio* of the transformer?

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