Lecture 29: WED 25 MAR
Magnetic Fields Due to Currents II

Force Between Two Parallel Currents
Ampere’s Law

Jean-Baptiste Biot (1774-1862)
Felix Savart (1791–1841)
Electric Current: A Source of Magnetic Field

• Observation: an electric current creates a magnetic field
• Simple experiment: hold a current-carrying wire near a compass needle!
New Right Hand Rule!

- Point your thumb along the direction of the current in a straight wire.
- The magnetic field created by the current consists of circular loops directed along your curled fingers.
- The magnetic field gets weaker with distance: For long wire it’s a 1/R Law!
- You can apply this to ANY straight wire (even a small differential element!)
- What if you have a curved wire? Break into small elements.

\[ B \]

Direction of \( B \)!
Field of a Straight Wire

\[ d\vec{B} = \frac{\mu_0}{4\pi} \frac{id\vec{s} \times \vec{r}}{r^3} \]

\[ dB = \frac{\mu_0}{4\pi} \frac{id(s(r \sin \theta))}{r^3} \]

\[ \sin \theta = \frac{R}{r} \]

\[ r = (s^2 + R^2)^{1/2} \]

\[ B = \frac{\mu_0 i}{4\pi} \int_{-\infty}^{\infty} ds (r \sin \theta) \]

\[ = \frac{\mu_0 i}{4\pi} \int_{-\infty}^{\infty} \frac{Rds}{\left(s^2 + R^2\right)^{3/2}} \]

\[ = \frac{\mu_0 i}{2\pi} \int_{0}^{\infty} \frac{Rds}{\left(s^2 + R^2\right)^{3/2}} \]

\[ = \mu_0 i R \left[ \frac{s}{R^2 \left(s^2 + R^2\right)^{1/2}} \right]_0^\infty \]

\[ = \frac{\mu_0 i}{2\pi R} \]
**Biot-Savart Law**

- A circular arc of wire of radius $R$ carries a current $i$.
- What is the magnetic field at the center of the loop?

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{id\vec{s} \times \vec{r}}{r^3}$$

$$dB = \frac{\mu_0}{4\pi} \frac{id\vec{s}R}{R^3} = \frac{\mu_0}{4\pi} \frac{iRd\phi}{R^2}$$

$$B = \frac{\mu_0}{4\pi} \int \frac{id\phi}{R} = \frac{\mu_0 i\Phi}{4\pi R}$$

Direction of $B$?? Not another right hand rule?!

TWO right hand rules!:
If your thumb points along the CURRENT, your fingers will point in the same direction as the FIELD.

If you curl our fingers around direction of CURRENT, your thumb points along FIELD!
Magnetic field due to wire 1 where the wire 2 is,

\[ B_1 = \frac{\mu_0 I_1}{2\pi d} \]

Force on wire 2 due to this field,

\[ F_{21} = L I_2 B_1 = \frac{\mu_0 L I_1 I_2}{2\pi d} \]

eHarmony’s Rule for Currents: Same Currents – Attract! Opposite Currents – Repel!
29.3: Force Between Two Parallel Wires:

\[ B_a = \frac{\mu_0 i_a}{2\pi d}. \]

\[ \vec{F}_{ba} = i_b \vec{L} \times \vec{B}_a, \]

\[ F_{ba} = i_b L B_a \sin 90^\circ = \frac{\mu_0 L i_a i_b}{2\pi d}. \]

The field due to \( a \) at the position of \( b \) creates a force on \( b \).

**Fig. 29-9** Two parallel wires carrying currents in the same direction attract each other. \( \vec{B}_a \) is the magnetic field at wire \( b \) produced by the current in wire \( a \). \( \vec{F}_{ba} \) is the resulting force acting on wire \( b \) because it carries current in \( \vec{B}_a \).

To find the force on a current-carrying wire due to a second current-carrying wire, first find the field due to the second wire at the site of the first wire. Then find the force on the first wire due to that field.

Parallel currents attract each other, and antiparallel currents repel each other.
CHECKPOINT 1

The figure here shows three long, straight, parallel, equally spaced wires with identical currents either into or out of the page. Rank the wires according to the magnitude of the force on each due to the currents in the other two wires, greatest first.

\[ F_{21} = \frac{\mu_0 L i_1 i_2}{2\pi d} = \text{Force on 2 due to 1.} \]

Like currents attract & Opposites Repel.

\[ \frac{1}{r} \Rightarrow \text{double the distance halve the force.} \]

\[ F^\text{net}_b > F^\text{net}_c > F^\text{net}_a \]
29.3.3. The drawing represents a device called Roget’s Spiral. A coil of wire hangs vertically and its windings are parallel to one another. One end of the coil is connected by a wire to a terminal of a battery. The other end of the coil is slightly submerged below the surface of a cup of mercury. Mercury is a liquid metal at room temperature. The bottom of the cup is also metallic and connected by a wire to a switch. A wire from the switch to the battery completes the circuit. What is the behavior of this circuit after the switch is closed?

a) When current flows in the circuit, the coils of the wire move apart and the wire is extended further into the mercury.

b) Nothing happens to the coil because there will not be a current in this circuit.

c) A current passes through the circuit until all of the mercury is boiled away.

d) When current flows in the circuit, the coils of the wire move together, causing the circuit to break at the surface of the mercury. The coil then extends and the process begins again when the circuit is once again complete.
29.3.3. The drawing represents a device called Roget’s Spiral. A coil of wire hangs vertically and its windings are parallel to one another. One end of the coil is connected by a wire to a terminal of a battery. The other end of the coil is slightly submerged below the surface of a cup of mercury. Mercury is a liquid metal at room temperature. The bottom of the cup is also metallic and connected by a wire to a switch. A wire from the switch to the battery completes the circuit. What is the behavior of this circuit after the switch is closed?

a) When current flows in the circuit, the coils of the wire move apart and the wire is extended further into the mercury.

b) Nothing happens to the coil because there will not be a current in this circuit.

c) A current passes through the circuit until all of the mercury is boiled away.

d) When current flows in the circuit, the coils of the wire move together, causing the circuit to break at the surface of the mercury. The coil then extends and the process begins again when the circuit is once again complete.
29.3: Force Between Two Parallel Wires, Rail Gun:

Fig. 29-10  (a) A rail gun, as a current $i$ is set up in it. The current rapidly causes the conducting fuse to vaporize. (b) The current produces a magnetic field $\vec{B}$ between the rails, and the field causes a force $\vec{F}$ to act on the conducting gas, which is part of the current path. The gas propels the projectile along the rails, launching it.
Remember Gauss Law for E-Fields?

Given an arbitrary closed surface, the electric flux through it is proportional to the charge enclosed by the surface.

\[
\Phi_E \equiv \oint \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}
\]
Gauss’ s Law for B-Fields!

No isolated magnetic poles! The magnetic flux through any closed “Gaussian surface” will be ZERO. This is one of the four “Maxwell’s equations”.

\[ \Phi_B = \oint \mathbf{B} \cdot d\mathbf{A} = 0 \]

There are no SINKS or SOURCES of B-Fields!

What Goes IN Must Come OUT!
No isolated north or south “monopoles”. 
ICPP
Rank the magnitude of the magnetic flux
\[ \Phi_B = \oint B \cdot dA \]
for the three surfaces, greatest first.

Magnetic Flux is always what
for a closed surface!? 

\[ \Phi_a = \Phi_b = \Phi_c = 0 \]
Ampere’s law: Closed Loops

\[ \oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enclosed}} \]

The circulation of B (the integral of B scalar ds) along an imaginary closed loop is proportional to the net amount of current traversing the loop.

\[ \oint \vec{B} \cdot d\vec{s} = \mu_0 (i_1 + i_2 - i_3) \]

Thumb rule for sign; ignore \( i_4 \)

If you have a lot of symmetry, knowing the circulation of B allows you to know B.
Ampere’s law: Closed Loops

\[ \oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enclosed}} \]

The circulation of \( B \) (the integral of \( B \) scalar \( ds \)) along an imaginary **closed loop** is proportional to the net amount of current piercing the loop.

\[ \oint \vec{B} \cdot d\vec{s} = \mu_0 (i_1 - i_2) \]

Thumb rule for sign; ignore \( i_3 \)

If you have a lot of **symmetry**, knowing the circulation of \( B \) allows you to know \( B \).
Calculation of $i_{\text{enc}}$. We curl the fingers of the right hand in the direction in which the Amperian loop was traversed. We note the direction of the thumb.

All currents inside the loop **parallel** to the thumb are counted as **positive**. All currents inside the loop **antiparallel** to the thumb are counted as **negative**. All currents outside the loop are not counted.

In this example: $i_{\text{enc}} = i_1 - i_2$. 
ICPP:

- Two square conducting loops carry currents of 5.0 and 3.0 A as shown in Fig. 30-60. What’s the value of $\int \mathbf{B} \cdot d\mathbf{s}$ through each of the paths shown?

Path 1: $\int \mathbf{B} \cdot d\mathbf{s} = \mu_0(-5.0A + 3.0A)$

Path 2: $\int \mathbf{B} \cdot d\mathbf{s} = \mu_0(-5.0A - 5.0A - 3.0A)$
CHECKPOINT 2

The figure here shows three equal currents $i$ (two parallel and one antiparallel) and four Amperian loops. Rank the loops according to the magnitude of $\oint \overrightarrow{B} \cdot d\overrightarrow{s}$ along each, greatest first.

\begin{align*}
(a) & \ -i + i + i = i \\
(b) & \ -i + 0 + i = 0 \\
(c) & \ -i + 0 + 0 = -i \\
(d) & \ 0 + i + i = 2i \\
\end{align*}

$d > a = c > b = 0$
Ampere’s Law: Example 1

- Infinitely long straight wire with current \( i \).
- Symmetry: magnetic field consists of circular loops centered around wire.
- So: choose a circular loop \( C \) so \( B \) is tangential to the loop everywhere!
- Angle between \( B \) and \( ds \) is 0. (Go around loop in same direction as \( B \) field lines!)

\[
\oint_C \vec{B} \cdot d\vec{s} = \mu_0 i
\]

\[
\oint_C B ds = B(2\pi R) = \mu_0 i
\]

Much Easier Way to Get B-Field Around A Wire: No Calculus!

\[
B = \frac{\mu_0 i}{2\pi R}
\]
29.4: Ampere’s Law, Magnetic Field Outside a Long Straight Wire Carrying Current:

All of the current is encircled and thus all is used in Ampere's law.

\[ \oint \vec{B} \cdot d\vec{s} = \oint B \cos \theta \, ds = B \oint ds = B(2\pi r). \]

\[ B(2\pi r) = \mu_0 i \]

\[ B = \frac{\mu_0 i}{2\pi r} \] (outside straight wire).

**Fig. 29-13** Using Ampere’s law to find the magnetic field that a current \( i \) produces outside a long straight wire of circular cross section. The Amperian loop is a concentric circle that lies outside the wire.
Ampere’s Law: Example 2

- Infinitely long cylindrical wire of finite radius $R$ carries a total current $i$ with uniform current density.
- Compute the magnetic field at a distance $r$ from cylinder axis for:
  - $r < a$ (inside the wire)
  - $r > a$ (outside the wire)

\[ \oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 i \]
Ampere’s Law: Example 2 (cont)

\[ \oint_C \vec{B} \cdot d\vec{s} = \mu_0 I \]

\[ B(2\pi r) = \mu_0 I_{\text{enclosed}} \]

\[ B = \frac{\mu_0 I_{\text{enclosed}}}{2\pi r} \]

\[ i_{\text{enclosed}} = J(\pi r^2) = \frac{i}{\pi R^2} \pi r^2 = i \frac{r^2}{R^2} \]

\[ B = \frac{\mu_0 i r}{2\pi R^2} \quad \text{For } r < R \]

For \( r > R \), \( i_{\text{enclosed}} = i \), so \( B = \frac{\mu_0 i}{2\pi R} = \text{LONG WIRE!} \)
B-Field In/Out Wire: J is Constant

\[ B = \frac{\mu_0 i r}{2\pi R^2} \]

For \( r < R \)

\[ B \propto r \]

For \( r > R \)

\[ B \propto \frac{1}{r} \]

Outside Long Wire!
Example, Ampere’s Law to find the magnetic field inside a long cylinder of current when J is NOT constant. Must integrate!

Figure 29-15a shows the cross section of a long conducting cylinder with inner radius \( a = 2.0 \text{ cm} \) and outer radius \( b = 4.0 \text{ cm} \). The cylinder carries a current out of the page, and the magnitude of the current density in the cross section is given by \( J = cr^2 \), with \( c = 3.0 \times 10^6 \text{ A/m}^4 \) and \( r \) in meters. What is the magnetic field \( \vec{B} \) at the dot in Fig. 29-15a, which is at radius \( r = 3.0 \text{ cm} \) from the central axis of the cylinder?

\[ \int \vec{B} \cdot d\vec{S} = \mu_0 i_{enc}, \]

gives us

\[ B(2\pi r) = -\frac{\mu_0 \pi c}{2} (r^4 - a^4). \]

Solving for \( B \) and substituting known data yield

\[ B = -\frac{\mu_0 c}{4r} (r^4 - a^4) \]

\[ = -\frac{(4\pi \times 10^{-7} \text{ T} \cdot \text{m/A})(3.0 \times 10^6 \text{ A/m}^4)}{4(0.030 \text{ m})} \times [(0.030 \text{ m})^4 - (0.020 \text{ m})^4] \]

\[ = -2.0 \times 10^{-5} \text{ T}. \]

Thus, the magnetic field \( \vec{B} \) at a point 3.0 cm from the central axis has magnitude

\[ B = 2.0 \times 10^{-5} \text{ T} \] (Answer)

**Calculations:** We write the integral as

\[ i_{enc} = \int J \, dA = \int_a^r cr^2(2\pi r \, dr) \]

\[ = 2\pi c \int_a^r r^3 \, dr = 2\pi c \left[ \frac{r^4}{4} \right]_a^r \]

\[ = \frac{\pi c(r^4 - a^4)}{2}. \]