Physics 2113
Lecture 03: WED 17 JAN
CH13: Gravitation III

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Version: 12/26/14
13.7: Planets and Satellites: Kepler’s 1st Law

1. THE LAW OF ORBITS: All planets move in elliptical orbits, with the Sun at one focus.

Laws were based on data fits!

The Sun is at one of the two focal points.

Fig. 13-12 A planet of mass $m$ moving in an elliptical orbit around the Sun. The Sun, of mass $M$, is at one focus $F$ of the ellipse. The other focus is $F'$, which is located in empty space. Each focus is a distance $ea$ from the ellipse’s center, with $e$ being the eccentricity of the ellipse. The semimajor axis $a$ of the ellipse, the perihelion (nearest the Sun) distance $R_p$, and the aphelion (farthest from the Sun) distance $R_a$ are also shown.

Tycho Brahe 1546–1601

Johannes Kepler 1571–1630
2. THE LAW OF AREAS:
A line that connects a planet to the Sun sweeps out equal areas in the plane of the planet’s orbit in equal time intervals; that is, the rate $dA/dt$ at which it sweeps out area $A$ is constant.

Angular momentum, $L$:

$$L = rp_\perp = (r)(mv_\perp) = (r)(mωr) = mr^2ω,$$

$$\frac{dA}{dt} = \frac{1}{2}r^2 \frac{dθ}{dt} = \frac{1}{2}r^2ω,$$

$$\Rightarrow A \propto t$$
3. THE LAW OF PERIODS: The square of the period of any planet is proportional to the cube of the semi-major axis of its orbit.

Consider a circular orbit with radius $r$ (the radius of a circle is equivalent to the semimajor axis of an ellipse). Applying Newton’s second law to the orbiting planet yields

$$\frac{GMm}{r^2} = (m)(\omega^2 r).$$

Using the relation of the angular velocity, $\omega$, and the period, $T$, one gets:

$$T = \frac{2\pi}{\omega}.$$

$$T^2 = \left(\frac{4\pi^2}{GM}\right)r^3 \quad \text{(law of periods).}$$

<table>
<thead>
<tr>
<th>Planet</th>
<th>Semimajor Axis $a$ ($10^{10}$ m)</th>
<th>Period $T$ (y)</th>
<th>$T^2/a^3$ ($10^{-34}$ y²/m³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>5.79</td>
<td>0.241</td>
<td>2.99</td>
</tr>
<tr>
<td>Venus</td>
<td>10.8</td>
<td>0.615</td>
<td>3.00</td>
</tr>
<tr>
<td>Earth</td>
<td>15.0</td>
<td>1.00</td>
<td>2.96</td>
</tr>
<tr>
<td>Mars</td>
<td>22.8</td>
<td>1.88</td>
<td>2.98</td>
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<tr>
<td>Jupiter</td>
<td>77.8</td>
<td>11.9</td>
<td>3.01</td>
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<tr>
<td>Saturn</td>
<td>143</td>
<td>29.5</td>
<td>2.98</td>
</tr>
<tr>
<td>Uranus</td>
<td>287</td>
<td>84.0</td>
<td>2.98</td>
</tr>
<tr>
<td>Neptune</td>
<td>450</td>
<td>165</td>
<td>2.99</td>
</tr>
<tr>
<td>Pluto</td>
<td>590</td>
<td>248</td>
<td>2.99</td>
</tr>
</tbody>
</table>
CHECKPOINT 4

Satellite 1 is in a certain circular orbit around a planet, while satellite 2 is in a larger circular orbit. Which satellite has (a) the longer period and (b) the greater speed?

(a) The larger the orbit the longer the period: SAT-2.

(b) The smaller the orbit the greater the speed: SAT-1.

\[
T = \sqrt{\frac{4\pi^2 r^3}{GM}} \propto r^{3/2}
\]

\[
v = \omega r = \frac{2\pi}{T} = \sqrt{\frac{GM}{r}} \propto \frac{1}{\sqrt{r}}
\]

\[
r_{\text{LEO}} = R_{\text{Earth}} + a_{\text{LEO}} = 10^7 \text{ m}
\]

\[
T_{\text{LEO}} = \sqrt{\frac{4\pi^2}{\frac{1}{6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}}} \frac{1}{5.97 \times 10^{24} \text{ kg}}} \frac{(10^7 \text{ m})^3}{1} = 6307 \text{ s}
\]

\[
v_{\text{LEO}} = \sqrt{\frac{6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}}{1}} \frac{1}{5.97 \times 10^{24} \text{ kg}} \frac{1}{10^7 \text{ m}} = 7.4 \text{ km/s}
\]

\[
r_{\text{GEO}} = R_{\text{Earth}} + a_{\text{GEO}} = 4.22 \times 10^7 \text{ m}
\]

\[
T_{\text{GEO}} = \sqrt{\frac{4\pi^2}{\frac{1}{6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}}} \frac{1}{5.97 \times 10^{24} \text{ kg}}} \frac{(4.22 \times 10^7 \text{ m})^3}{1} = 8.62 \times 10^4 \text{ s} = 24 \text{ hrs}
\]

\[
v_{\text{GEO}} = \sqrt{\frac{6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}}{1}} \frac{1}{5.97 \times 10^{24} \text{ kg}} \frac{1}{4.22 \times 10^7 \text{ m}} = 3 \text{ km/s}
\]
13.7: Newton Derived Kepler’s Laws from Inverse Square Law!
http://galileo.phys.virginia.edu/classes/152.mf1i.spring02/KeplersLaws.htm

Kepler’s Second Law First: Equal Areas Proportional to Equal Time!

\[ v_r = dr/dt, \quad v_\perp = rd\theta / dt = r\omega \quad \omega = d\theta / dt \]

Area swept out in small \( \Delta t \) \( \approx \frac{1}{2} r^2 \Delta \theta = \frac{1}{2} r^2 \frac{\Delta \theta}{\Delta t} \Delta t = \frac{1}{2} r^2 \omega \Delta t \)

\[ L = mrv_\perp = mr^2 \omega \quad \text{Angular Momentum} \]

Rate of sweeping out of area,
\[ dA / dt = c \]
is proportional to the angular momentum \( L \), and equal to \( L/2m = \text{Constant} = C \).

\[ \Rightarrow A \propto t \]
13.7: Newton Derived Kepler’s Laws from Inverse Square Law!

http://galileo.phys.virginia.edu/classes/152.mf1i.spring02/KeplersLaws.htm

Kepler’s First Law:
Ellipses with Sun at Focus

\( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)

\( \frac{a(1-e^2)}{r} = 1 + e \cos \theta \)

\( \frac{d^2r}{dt^2} = \frac{GM}{r^2} - r \omega^2 = -\frac{GM}{r^2} \)

\( \omega = \frac{mr^2}{L} \)

\( L = mr^2 \omega = \frac{m}{u^2} \frac{d\theta}{dt} \)

\( \frac{d^2u}{d\theta^2} + u = \frac{GMM^2}{L^2} \)

This is equivalent to the standard \((r, q)\) equation of an ellipse of semi-major axis \(a\) and eccentricity \(e\), with the origin — the Sun — at one focus. Note \(1/L^2\) is from inverse Square Law.
Kepler's 3rd Law:
For Ellipse

\[ T = \frac{\pi ab}{L/2m} \]

\[ \frac{1}{r} = \frac{GMm^2}{L^2} + Acos\theta \]

\[ \frac{a(1-e^2)}{r} = 1+e\cos\theta \]

\[ \frac{L^2}{GMm^2} = a(1-e^2) \]

\[ b^2 = a^2 \left(1-e^2\right) \]

\[ T^2 = \left(2m\pi ab\right)^2 / L^2 \]

\[ = \left(2m\pi ab\right)^2 / GMm^2 a \left(1-e^2\right) \]

\[ = \left(2m\pi ab\right)^2 / GMm^2 a \left(b^2 / a^2\right) \]

\[ = 4\pi^2 a^3 / GM. \]

\[ T^2 \propto a^3 \]
Example, Halley’s Comet

Comet Halley orbits the Sun with a period of 76 years and, in 1986, had a distance of closest approach to the Sun, its perihelion distance \( R_p \), of \( 8.9 \times 10^{10} \) m. Table 13-3 shows that this is between the orbits of Mercury and Venus.

(a) What is the comet’s farthest distance from the Sun, which is called its aphelion distance \( R_a \)?

**KEY IDEAS**

From Fig. 13-12, we see that \( R_a + R_p = 2a \), where \( a \) is the semimajor axis of the orbit. Thus, we can find \( R_a \) if we first find \( a \). We can relate \( a \) to the given period via the law of periods (Eq. 13-34) if we simply substitute the semimajor axis \( a \) for \( r \).

**Calculations:** Making that substitution and then solving for \( a \), we have

\[
a = \left( \frac{GMT^2}{4\pi^2} \right)^{1/3}.
\]  

(13-35)

If we substitute the mass \( M \) of the Sun, \( 1.99 \times 10^{30} \) kg, and the period \( T \) of the comet, 76 years or \( 2.4 \times 10^9 \) s, into Eq. 13-35, we find that \( a = 2.7 \times 10^{12} \) m. Now we have

\[
R_a = 2a - R_p
= (2)(2.7 \times 10^{12} \text{ m}) - 8.9 \times 10^{10} \text{ m}
= 5.3 \times 10^{12} \text{ m}.
\]  

(Answer)

Table 13-3 shows that this is a little less than the semimajor axis of the orbit of Pluto. Thus, the comet does not get farther from the Sun than Pluto.

(b) What is the eccentricity \( e \) of the orbit of comet Halley?

**KEY IDEA**

We can relate \( e, a, \) and \( R_p \) via Fig. 13-12, in which we see that \( ea = a - R_p \).

**Calculation:** We have

\[
e = \frac{a - R_p}{a} = 1 - \frac{R_p}{a}
\]  

(13-36)

\[
e = 1 - \frac{8.9 \times 10^{10} \text{ m}}{2.7 \times 10^{12} \text{ m}} = 0.97.
\]  

(Answer)

This tells us that, with an eccentricity approaching unity, this orbit must be a long thin ellipse.
13.8: Satellites: Orbits and Energy

As a satellite orbits Earth in an elliptical path, the mechanical energy $E$ of the satellite remains constant. Assume that the satellite’s mass is so much smaller than Earth’s mass.

The potential energy of the system is given by

$$U = -\frac{GMm}{r}$$

For a satellite in a circular orbit,

$$\frac{GMm}{r^2} = m \frac{v^2}{r},$$

Thus, one gets:

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r}$$

$$E = -\frac{GMm}{2r} \quad \text{(circular orbit)}.$$ 

For an elliptical orbit (semimajor axis $a$),

$$E = -\frac{GMm}{2a}$$
In the figure here, a space shuttle is initially in a circular orbit of radius $r$ about Earth. At point $P$, the pilot briefly fires a forward-pointing thruster to decrease the shuttle’s kinetic energy $K$ and mechanical energy $E$. (a) Which of the dashed elliptical orbits shown in the figure will the shuttle then take? (b) Is the orbital period $T$ of the shuttle (the time to return to $P$) then greater than, less than, or the same as in the circular orbit?

(a) path 1: As $E$ decreases ($dE < 0$); $r$ decreases ($dr < 0$)

(b) Less: As $r$ decreases ($dr < 0$); $T$ decreases ($dT < 0$)
Earth is NOT a Uniform Sphere —> Gravitational Field Changes in Orbit.

\[ g = \frac{GM}{r^2} \propto + \frac{1}{r^2} \]
\[ dg \propto -\frac{1}{r^3} \, dr \]

As \( g \) increases (\( dg > 0 \)); \( r \) decreases (\( dr < 0 \)).

\[ v = \sqrt{\frac{GM}{r}} \propto + r^{-\frac{1}{2}} \]
\[ dv \propto -r^{-\frac{3}{2}} \, dr \]

As \( r \) decreases (\( dr < 0 \)); \( v \) increases (\( dv > 0 \)).

- Changing field \( \Delta g \) give rise to changing velocity \( \Delta v \).
- Changing \( \Delta v \) gives changing satellite-to-satellite distance.
- Microwave link measures changing distance between satellites.
- Measuring \( \Delta g \) allows computation of \( \Delta M \) — Earth’s Mass Distribution.
Example, Mechanical Energy of a Bowling Ball

A playful astronaut releases a bowling ball, of mass \( m = 7.20 \text{ kg} \), into circular orbit about Earth at an altitude \( h \) of 350 km.

(a) What is the mechanical energy \( E \) of the ball in its orbit?

We can get \( E \) from the orbital energy, given by Eq. 13-40 \((E = -\frac{GMm}{2r})\), if we first find the orbital radius \( r \) : (It is not simply the given altitude.)

**Calculations:** The orbital radius must be

\[ r = R + h = 6370 \text{ km} + 350 \text{ km} = 6.72 \times 10^6 \text{ m}, \]

in which \( R \) is the radius of Earth. Then, from Eq. 13-40, the mechanical energy is

\[
E = -\frac{GMm}{2r} = -\frac{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)(5.98 \times 10^{24} \text{ kg})(7.20 \text{ kg})}{(2)(6.72 \times 10^6 \text{ m})} = -2.14 \times 10^8 \text{ J} = -214 \text{ MJ}. \]

(Answer)

(b) What is the mechanical energy \( E_0 \) of the ball on the launchpad at Cape Canaveral (before it, the astronaut, and the spacecraft are launched)? From there to the orbit, what is the change \( \Delta E \) in the ball’s mechanical energy?

On the launchpad, the ball is not in orbit and thus Eq. 13-40 does not apply. Instead, we must find \( E_0 = K_0 + U_0 \), where \( K_0 \) is the ball’s kinetic energy and \( U_0 \) is the gravitational potential energy of the ball–Earth system.

**Calculations:** To find \( U_0 \), we use Eq. 13-21 to write

\[
U_0 = -\frac{GMm}{R} = -\frac{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)(5.98 \times 10^{24} \text{ kg})(7.20 \text{ kg})}{6.37 \times 10^6 \text{ m}} = -4.51 \times 10^8 \text{ J} = -451 \text{ MJ}. \]

The kinetic energy \( K_0 \) of the ball is due to the ball’s motion with Earth’s rotation. You can show that \( K_0 \) is less than 1 MJ, which is negligible relative to \( U_0 \). Thus, the mechanical energy of the ball on the launchpad is

\[
E_0 = K_0 + U_0 \approx 0 - 451 \text{ MJ} = -451 \text{ MJ}. \quad \text{(Answer)}
\]

The increase in the mechanical energy of the ball from launchpad to orbit is

\[
\Delta E = E - E_0 = (-214 \text{ MJ}) - (-451 \text{ MJ}) = 237 \text{ MJ}. \quad \text{(Answer)}
\]

This is worth a few dollars at your utility company. Obviously the high cost of placing objects into orbit is not due to their required mechanical energy.
13.9: Einstein and Gravitation: Curvature of Space

Fig. 13-18 (a) Two objects moving along lines of longitude toward the south pole converge because of the curvature of Earth’s surface. (b) Two objects falling freely near Earth move along lines that converge toward the center of Earth because of the curvature of space near Earth. (c) Far from Earth (and other masses), space is flat and parallel paths remain parallel. Close to Earth, the parallel paths begin to converge because space is curved by Earth’s mass.
13.9: Einstein and Gravitation: Gravity Waves

**THE SEARCH FOR GRAVITY WAVES**

Gravity is one of the universe's basic forces. It gives weight to objects with mass. According to Isaac Newton, the force by which gravity attracts two bodies is proportional to their mass. However, in 1915 Albert Einstein suggested a different explanation. The effects of gravity occurred because objects with mass bend the fabric of space, known as spacetime, so that free-falling objects find their paths curved or deflected.

**Einstein's theory**

In his theory of general relativity, Einstein argued that the motion of an object would cause ripples to emanate through the curvature of space-time. These fluctuations are known as gravitational waves, shown here radiating from a binary star system – two ultra-dense neutron stars that are spiraling closer and closer to each other.

**Gravitational waves**

Disturbances in the Gravitational Field Move Outward As Waves

**Wave detector**

The wave detectors work by a process of firing and reflecting laser beams across two arms, giving the device its distinctive L-shape.

**Laser**

The beams bounce back down the arms after being reflected by mirrors.

**Beam splitter**

Laser light is projected into a beam splitter. The resulting two beams are projected down the two arms of the gravitational wave detector.

**Photometers**

Change in lengths of the arms caused by gravitational waves will change light signals when beams are recombined.

**LIVINGSTON LASER INTERFEROMETER GRAVITATIONAL-WAVE OBSERVATORY**

Two Orbiting Black Holes
HW01 DUE TONIGHT: 11:59PM FRI 29 AUG!
WEB ASSIGN CLASS KEY FOR SECTION 2: lsu 8181 3713

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