A uniform shell of matter exerts no net gravitational force on a particle located inside it.

The components of the force in the x-direction cancel out by symmetry.

The components of the net force in the z-direction add up by symmetry.

The total net force integrates up to zero.

Proof same as for \( r > R \) but with different limits of integration!

Applying the shell law to concentric shells proves can treat Earth (uniform sphere) as if all mass are in shells \( \leq r \) are at center and NO shells with radius \( > r \) contribute any force at all!

\[
F_{\text{net}} = \frac{GMm}{4r^2R} \int_{R-r}^{R+r} \left(1 + \frac{r^2 - R^2}{s^2} \right) ds = 0
\]
13.5: Gravitation Inside Earth: Shell Game II

1. A uniform shell of matter exerts no net gravitational force on a particle located inside it.

2. A uniform shell of matter exerts a force on a particle located outside it as if all the mass was at the center.

\[
\text{density} = \rho = \frac{M_{\text{tot}}}{V_{\text{tot}}}
\]

\[
M_{\text{ins}} = \rho V_{\text{ins}} = \frac{M}{V_{\text{tot}}} V_{\text{ins}} = M \frac{r^3}{R^3}
\]

\[
\text{force} = F = \frac{GmM_{\text{ins}}}{r^2} = \frac{Gm}{r^2} \left( M \frac{r^3}{R^3} \right) = \frac{GmM}{R^3} r
\]

\[
\text{field} = g = \frac{GM_{\text{ins}}}{r^2} = \frac{G}{r^2} \left( M \frac{r^3}{R^3} \right) = \frac{GM}{R^3} r
\]

Inside the Earth the Force and Field Scale LINEARLY with \( r \). This is like Hooke’s Law for a Mass on a Spring.
13.5: Gravitation Inside Earth: Summary Moving From Center Out

1. INSIDE a uniform sphere field/force INCREASES like $r$
2. OUTSIDE a uniform sphere field/force DECREASES like $1/r^2$
13.5: Gravitation Inside Earth: Gauss’s Law for Gravity


Case I: \( r > R \)

\[
M_{\text{ins}} = M \\
g(r)4\pi r^2 = -4\pi GM \\
g(r) = -\frac{GM}{r^2} \quad \sqrt{ }
\]

Case II: \( r < R \)

\[
M_{\text{ins}} = M \frac{r^3}{R^3} \\
g(r)4\pi r^2 = -4\pi GM \frac{r^3}{R^3} \\
g(r) = -\frac{GM}{R^3}r \quad \sqrt{ }
\]

\[
\int_S \vec{g} \cdot d\vec{S} = -4\pi GM_{\text{ins}} \quad \text{Gauss's Law} \\
\vec{g} = g(r)\hat{n} \\
\int_S \vec{g} \cdot d\vec{S} = \int_S \vec{g} \cdot \hat{n}dS = g(r)\int_S dS = g(r)4\pi r^2
\]
13.5: Gravitation Inside Earth: Gauss’s Law for Gravity

ICPP: Evaluate the Surface Integral For the Two Gaussian Surfaces S_I & S_{II}.

\[ \int_{S_I} \mathbf{g} \cdot d\mathbf{S} = -4\pi G M_{\text{ins}} \]
\[ = -4\pi G M_{\text{Earth}} \]

\[ \int_{S_{II}} \mathbf{g} \cdot d\mathbf{S} = -4\pi G M_{\text{ins}} \]
\[ = 0 \]
13.5: Gravitation Inside Earth

A uniform shell of matter exerts no net gravitational force on a particle located inside it.

Sample Problem

In the 2012 remake of the film *Total Recall*, Colin Farrell rides a train that falls through the center of the Earth.

In the film Farrell experiences normal gravity until he hits the core, then experiences a moment of weightlessness at the core, and then resumes normal gravity (in the opposite direction) as the train continues to the other side of the Earth.

Decide if this is what really would happen (or if it is complete Hollywood BS) by finding the gravitational force on the capsule of mass $m$ when it reaches a distance $r$ from Earth’s center. Assume that Earth is a sphere of uniform density $\rho$ (mass per unit volume).

Calculations:

The force magnitude depends linearly on the capsule’s distance $r$ from Earth’s center. Thus, as $r$ decreases, $F$ also decreases, until it is zero at Earth’s center.

However the train and occupants are both in free fall would be weightless the entire time! Complete Hollywood BS!
13.6: Gravitational Potential Energy

The gravitational potential energy of the two-particle system is:

\[ U = -\frac{GMm}{r} \]

\( U(r) \) approaches zero as \( r \) approaches infinity and that for any finite value of \( r \), the value of \( U(r) \) is negative.

If the system contains more than two particles, consider each pair of particles in turn, calculate the gravitational potential energy of that pair with the above relation, as if the other particles were not there, and then algebraically sum the results. That is,

\[ U = \left( -\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}} \right). \]

ICPP: If all the \( m \)'s and \( r \)'s the same how would you calculate the velocity of the masses the moment they are set loose?

\[ U = 3\left( \frac{Gm^2}{r} \right) = 3 \frac{1}{2} mv^2 = 3KE \]

\[ v = \sqrt{\frac{Gm}{2r}} \] units: \[ \sqrt{\frac{m^3 \text{ kg}}{\text{kg} \cdot \text{s}^2 \cdot \text{m}}} = \sqrt{\frac{m^2}{\text{s}^2}} = \frac{m}{\text{s}} \]
Let us shoot a baseball directly away from Earth along the path in the figure. We want to find the gravitational potential energy $U$ of the ball at point $P$ along its path, at radial distance $R$ from Earth’s center. The work $W$ done on the ball by the gravitational force as the ball travels from point $P$ to a great (infinite) distance from Earth is:

$$W = \int_{R}^{\infty} \vec{F}(r) \cdot d\vec{r}.$$ 

$$\vec{F}(r) \cdot d\vec{r} = F(r) \, dr \cos \phi = -\frac{G M m}{r^2} \, dr.$$ 

$$W = -G M m \int_{R}^{\infty} \frac{1}{r^2} \, dr = \left[ -\frac{G M m}{r} \right]_{R}^{\infty} = 0 - \frac{G M m}{R} = -\frac{G M m}{R},$$

where $W$ is the work required to move the ball from point $P$ (at distance $R$) to infinity. Work can also be expressed in terms of potential energies as

$$U_{\infty} - U = -W.$$ 

$$U = W = -\frac{G M m}{R}.$$
Gravitational Potential Energy $U$ vs. Gravitational Potential $V$

**Gravitational Potential Energy**:  
$U = -\frac{GmM}{r}$  
(Units: Joules = J = kg•m$^2$/s$^2$)

**Gravitational Potential**:  
$V = -\frac{GM}{r}$  
(Units: J/kg = m$^2$/s$^2$)

**Given the Potential Energy, Find the Potential**:

**Given the Potential, Find the Field**:

- $U = mV$
- $g = -\frac{dV}{dr}$

**Equal Potential Lines Perpendicular to Field Lines**

Note: Potential Exists in Empty Space Whether Test Mass $m$ is There or Not!
13.6: Gravitational Potential Energy
Path Independence

The work done along each circular arc is zero, because the direction of $F$ is perpendicular to the arc at every point. Thus, $W$ is the sum of only the works done by $F$ along the three radial lengths.

The gravitational force is a conservative force. Thus, the work done by the gravitational force on a particle moving from an initial point $i$ to a final point $f$ is independent of the path taken between the points. The change $\Delta U$ in the gravitational potential energy from point $i$ to point $f$ is given by

$$\Delta U = U_f - U_i = -W.$$

Since the work $W$ done by a conservative force is independent of the actual path taken, the change $\Delta U$ in gravitational potential energy is also independent of the path taken.
13.6: Gravitational Potential Energy

Path Independence

\[ U = mgh = 200\text{kg} \times 10 \frac{\text{m}}{\text{s}^2} \times h \]

\[
\begin{align*}
U_6 &= 12,000\text{J} \\
U_5 &= 10,000\text{J} \\
U_4 &= 8000\text{J} \\
U_3 &= 6000\text{J} \\
U_2 &= 4000\text{J} \\
U_1 &= 2000\text{J} \\
U_0 &= 0
\end{align*}
\]

ICPP: Mike takes the path shown. What is change in potential energy \( \Delta U \)?

The derivative of the potential energy $U$ gives the force $F$. The minus sign indicates that the force on mass $m$ points radially inward, toward mass $M$. 

\[ (F(x) = -\frac{dU(x)}{dx}) \]

\[ F = -\frac{dU}{dr} = -\frac{d}{dr} \left( -\frac{GMm}{r} \right) \]

\[ = -\frac{GMm}{r^2}. \]
13.6: Gravitational Potential Energy: Potential and Field

The derivative of the potential $V$ gives the field $g$.

The minus sign indicates that the field points radially inward, toward mass $M$.

$$
\mathbf{g} = -\nabla V \\
\mathbf{g} = -\frac{GM}{r^2}
$$
If you fire a projectile upward, there is a certain minimum initial speed that will cause it to move upward forever, theoretically coming to rest only at infinity.

This minimum initial speed is called the (Earth) escape speed.

Consider a projectile of mass $m$, leaving the surface of a planet (mass $M$, radius $R$) with escape speed $v$. The projectile has a kinetic energy $K$ given by $\frac{1}{2}mv^2$, and a potential energy $U$ given by:

$$U = -\frac{GMm}{R}$$

When the projectile reaches infinity, it stops and thus has no kinetic energy. It also has no potential energy because an infinite separation between two bodies is our zero-potential-energy configuration. Its total energy at infinity is therefore zero. From the principle of conservation of energy, its total energy at the planet’s surface must also have been zero, and so

$$K + U = \frac{1}{2}mv^2 + \left(-\frac{GMm}{R}\right) = 0.$$ 

This gives the escape speed

$$v = \sqrt{\frac{2GM}{R}}.$$
13.6: Gravitational Potential Energy: Escape Speed

\[ v = \sqrt{\frac{2GM}{R}}. \]

### Table 13-2

<table>
<thead>
<tr>
<th>Body</th>
<th>Mass (kg)</th>
<th>Radius (m)</th>
<th>Escape Speed (km/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ceres</td>
<td>$1.17 \times 10^{21}$</td>
<td>$3.8 \times 10^5$</td>
<td>0.64</td>
</tr>
<tr>
<td>Earth’s moon</td>
<td>$7.36 \times 10^{22}$</td>
<td>$1.74 \times 10^6$</td>
<td>2.38</td>
</tr>
<tr>
<td>Earth</td>
<td>$5.98 \times 10^{24}$</td>
<td>$6.37 \times 10^6$</td>
<td>11.2</td>
</tr>
<tr>
<td>Jupiter</td>
<td>$1.90 \times 10^{27}$</td>
<td>$7.15 \times 10^7$</td>
<td>59.5</td>
</tr>
<tr>
<td>Sun</td>
<td>$1.99 \times 10^{30}$</td>
<td>$6.96 \times 10^8$</td>
<td>618</td>
</tr>
<tr>
<td>Sirius B</td>
<td>$2 \times 10^{30}$</td>
<td>$1 \times 10^7$</td>
<td>5200</td>
</tr>
<tr>
<td>Neutron star</td>
<td>$2 \times 10^{30}$</td>
<td>$1 \times 10^4$</td>
<td>$2 \times 10^5$</td>
</tr>
</tbody>
</table>

*aThe most massive of the asteroids.
*b A white dwarf (a star in a final stage of evolution) that is a companion of the bright star Sirius.
*cThe collapsed core of a star that remains after that star has exploded in a supernova event.

ICPP: How Would You Find the Ratio \( M/R \) for a Black Hole Where \( v \geq c \), the speed of light?

\[ v = \sqrt{\frac{2GM}{R}} \geq c \quad \text{(Black Hole)} \]

\[
\frac{M}{R} \geq \frac{c^2}{2G} = 6.70 \times 10^{26} \frac{\text{kg}}{\text{m}}
\]

\[
M \equiv 8.2 \times 10^{36} \text{ kg} = 4.1 \text{ Million Solar Masses} \quad \text{(Mass of Object at Center of Galaxy)}
\]

\[
R \equiv 1.2 \times 10^{10} \text{ m} \quad \text{(Radius of Object at Center of Galaxy)}
\]

\[
\frac{M}{R} = 6.83 \times 10^{26} \quad \text{(Center of Galaxy is Super-Massive Black Hole!)}
\]
An asteroid, headed directly toward Earth, has a speed of 12 km/s relative to the planet when the asteroid is 10 Earth radii from Earth’s center. Neglecting the effects of Earth’s atmosphere on the asteroid, find the asteroid’s speed \( v_f \) when it reaches Earth’s surface.

**KEY IDEAS**

Because we are to neglect the effects of the atmosphere on the asteroid, the mechanical energy of the asteroid–Earth system is conserved during the fall. Thus, the final mechanical energy (when the asteroid reaches Earth’s surface) is equal to the initial mechanical energy. With kinetic energy \( K \) and gravitational potential energy \( U \), we can write this as

\[
K_f + U_f = K_i + U_i. \tag{13-29}
\]

Also, if we assume the system is isolated, the system’s linear momentum must be conserved during the fall. Therefore, the momentum change of the asteroid and that of Earth must be equal in magnitude and opposite in sign. However, because Earth’s mass is so much greater than the asteroid’s mass, the change in Earth’s speed is negligible relative to the change in the asteroid’s speed. So, the change in Earth’s kinetic energy is also negligible. Thus, we can assume that the kinetic energies in Eq. 13-29 are those of the asteroid alone.

**Calculations:** Let \( m \) represent the asteroid’s mass and \( M \) represent Earth’s mass \( (5.98 \times 10^{24} \text{ kg}) \). The asteroid is initially at distance \( 10R_E \) and finally at distance \( R_E \), where \( R_E \) is Earth’s radius \( (6.37 \times 10^6 \text{ m}) \). Substituting Eq. 13-21 for \( U \) and \( \frac{1}{2}mv^2 \) for \( K \), we rewrite Eq. 13-29 as

\[
\frac{1}{2}mv_f^2 - \frac{GMm}{R_E} = \frac{1}{2}mv_i^2 - \frac{GMm}{10R_E}.
\]

Rearranging and substituting known values, we find

\[
v_f^2 = v_i^2 + \frac{2GM}{R_E} \left( 1 - \frac{1}{10} \right)
= (12 \times 10^3 \text{ m/s})^2
+ \frac{2(6.67 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2)(5.98 \times 10^{24} \text{ kg})}{6.37 \times 10^6 \text{ m}} \times 0.9
= 2.567 \times 10^8 \text{ m}^2/\text{s}^2,
\]
and

\[
v_f = 1.60 \times 10^4 \text{ m/s} = 16 \text{ km/s}. \quad \text{(Answer)}
\]

At this speed, the asteroid would not have to be particularly large to do considerable damage at impact. If it were only 5 m across, the impact could release about as much energy as the nuclear explosion at Hiroshima. Alarminglly, about 500 million asteroids of this size are near Earth’s orbit, and in 1994 one of them apparently penetrated Earth’s atmosphere and exploded 20 km above the South Pacific (setting off nuclear-explosion warnings on six military satellites). The impact of an asteroid 500 m across (there may be a million of them near Earth’s orbit) could end modern civilization and almost eliminate humans worldwide.

In 2013 the **Chelyabinsk asteroid** moving at about this speed and weighing 10,000 metric tons = \( 10^8 \text{ kg} \) exploded over Russia. Estimate the energy released in Hiroshima units.

\[
K_f = \frac{1}{2} mv_f^2 \approx \frac{10^8 \text{ kg}}{2} \left( \frac{16 \times 10^3 \text{ m}}{\text{s}} \right)^2 \approx \frac{10^{16} \text{ kg} \cdot \text{m}^2}{\text{s}^2} = 10^{16} \text{ J} \frac{10^{16} \text{ J}}{10^{13} \text{ J} \cdot \text{H}} = 100 \text{H}
\]
You move a ball of mass $m$ away from a sphere of mass $M$. (a) Does the gravitational potential energy of the system of ball and sphere increase or decrease? (b) Is positive work or negative work done by the gravitational force between the ball and the sphere?

\[ U = -\frac{G M m}{r} \]  

(gravitational potential energy).

(a) $r$ increases

$\Rightarrow \frac{1}{r}$ decreases

$\Rightarrow -\frac{1}{r}$ increases

(b) YOU push ball uphill!

$\Rightarrow$ YOU do positive work!

$\Rightarrow$ FIELD does negative work!
\[
\frac{2m(1-x^2)}{2m} \sum_{n=0}^{m} P_n(x) y_n^L
\]

Oh, yeah!

\[
\sum_{n=0}^{\infty} T_n(x) y_n = y_n^{(H)} / B^2
\]